SOLUTION BY UNDETERMINED COEFFICIENTS

Let us consider the second order linear non homogeneous DE with constant coefficient.

$$y'' + ay' + by = f(x) \quad \text{——— (1)}$$

Since it is non homogeneous so its general solution is given by

$$y = y_c + y_p$$

y_c is the solution of homogeneous form of equation (1)

Calculation of y_p by method of undetermined Co-efficient

Table-1

Term in $f(x)$	choice of y_p
ke^{rx}	Ce^{rx}
kx^n $(n=0,1,2\cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

Rules of undetermined Co-efficient

Basic rule

If $f(x)$ is one of the functions of first column in table-1 then chose the y_p as the corresponding function from the 2nd column of the table-1

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Sum Rule:

If $f(x)$ is sum of the functions in the 1st column of table-1 then choose y_p as the sum of the corresponding functions from the 2nd column of table-1.

Modification rule

If your choice of y_p happens to be the solution of the homogeneous equation then replace your choice of y_p by multiplying $x, x^2 \ldots$ etc.

Some examples how to choose y_p corresponding to $f(x)$

Term in $f(x)$	choice of y_p
e^{4x}	Ce^{4x}
$2e^{-x}$	Ce^{-x}
5	K_0
$5+x$	$K_0 + K_1 x$
$5+x-2x^2$	$K_0 + K_1 x + K_2 x^2$
$2\cos x$	$K\cos x + M\sin x$
$\sin 3x$	$K\cos 3x + M\sin 3x$
$e^{-x}\cos 2x$	$e^{-x}(K\cos 2x + M\sin 2x)$
$2e^{4x}\sin x$	$e^{4x}(K\cos x + M\sin x)$
$2e^{x} - 5e^{-3x}$	$C_1 e^{x} + C_2 e^{-3x}$
$e^{4x} - x^2$	$Ce^{4x} + K_0 + K_1 x + K_2 x^2$
$e^{4x} + x^2$	$Ce^{4x} + K_0 + K_1 x + K_2 x^2$

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Term in f(x)	Choice of y_p
$e^x + 3\cos 2x$	$Ce^x + K\cos 2x + M\sin 2x$
$x - 2\sin x$	$K_0 + K_1 x + K\cos x + M\sin x$
$e^{-x} + e^{2x}\cos 3x$	$Ce^{-x} + e^{2x}(K\cos 3x + M\sin 3x)$
$x^2 - 2e^x \sin x$	$K_0 + K_1 x + K_2 x^2 + e^x(K\cos x + M\sin x)$

Example:-1

Solve $y'' + 4y = 8x^2$ ——— (1)

Ans:- The homogeneous form of equation (1) is

$$y'' + 4y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = 0$$

$$\Rightarrow (D^2 + 4)y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

The auxilary equation is

$$\lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 = -4 = 4i^2$$

$$\Rightarrow \lambda = \pm 2i$$

So, $y_c = e^{0x}(C_1\cos 2x + C_2 \sin 2x)$

$$\Rightarrow y_c = C_1\cos 2x + C_2 \sin 2x$$

Let $y_p = K_0 + K_1 x + K_2 x^2$

$$y_p' = K_1 + 2K_2 x$$

$$y_p'' = 2K_2$$

Putting these values in equation (1)

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$$2K_2 + 4K_0 + 4K_1 x + 4K_2 x^2 = 8x^2$$

Equating the like power of x of both sides

$$2K_2 + 4K_0 = 0 \Rightarrow 4 + 4K_0 = 0 \Rightarrow K_0 = -1$$

$$4K_1 = 0 \Rightarrow K_1 = 0$$

$$4K_2 = 8 \Rightarrow K_2 = 2$$

Thus $y_p = -1 + 0x + 2x^2 = 2x^2 - 1$

So, the general solution

$$y = y_c + y_p$$

$$\Rightarrow y = C_1 \cos 2x + C_2 \sin 2x + 2x^2 - 1$$

Example-2

Solve. $y'' - 3y' + 2y = e^{4x}$ ——— (1)

Ans:- The homogeneous form of equation (1) is

$$y'' - 3y' + 2y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$\Rightarrow (D^2 - 3D + 2)\, y = 0 \qquad \left(\because D = \frac{d}{dx}\right)$$

The auxilary equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 1, 2$$

So, $y_c = C_1 e^x + C_2 e^{2x}$

Let $y_p = Ce^{4x}$

$$y_p' = 4Ce^{4x}$$

$$y_p'' = 16Ce^{4x}$$

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Putting these values in equation (1)

$$16ce^{4x} - 12ce^{4x} + 2ce^{4x} = e^{4x}$$

$$\Rightarrow 6ce^{4x} = e^{4x}$$

$$\Rightarrow 6c = 1$$

$$\Rightarrow c = \frac{1}{6}$$

So, $y_p = \frac{1}{6} e^{4x}$

Hence the general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{6} e^{4x}.$$

Example-3

Solve, $y'' + 4y = \sin 3x$ ——— (1)

Ans:. The homogeneous form of equation (1) is

$$y'' + 4y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = 0$$

$$\Rightarrow (D^2 + 4)y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

The auxilary equation is

$$\lambda^2 + 4 = 0$$

$$\Rightarrow \lambda = \pm 2i$$

So, $y_c = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$$\Rightarrow y_c = c_1 \cos 2x + c_2 \sin 2x$$

Let $y_p = K \cos 3x + M \sin 3x$

$$y_p' = -3K \sin 3x + 3M \cos 3x$$

$$y_p'' = -9K \cos 3x - 9M \sin 3x$$

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Putting these values in equation (1)

$$-9K\cos 3x - 9M\sin 3x + 4K\cos 3x + 4M\sin 3x = \sin 3x$$

$$\Rightarrow -5K\cos 3x - 5M\sin 3x = \sin 3x$$

Equating both sides.

$$-5K = 0 \qquad -5M = 1$$

$$\Rightarrow K = 0 \qquad \Rightarrow M = -\frac{1}{5}$$

So, $y_p = 0\cos 3x - \frac{1}{5}\sin 3x = -\frac{1}{5}\sin 3x$

Thus the general solution is

$$y = y_c + y_p$$

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{1}{5}\sin 3x$$

Example-4

Solve. $y'' + 6y' + 9y = 50e^{-x}\cos x$ ——— (1)

Ans: The homogeneous form of equation (1) is

$$y'' + 6y' + 9y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

$$\Rightarrow (D^2 + 6D + 9)y = 0 \qquad \left(D = \frac{d}{dx}\right)$$

The auxiliary equation is

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda + 3)^2 = 0$$

$$\Rightarrow \lambda = -3, -3$$

So, $y_c = (c_1 + c_2x)e^{-3x}$

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Let $y_p = e^{-x}(K\cos x + M\sin x)$

$= Ke^{-x}\cos x + Me^{-x}\sin x$

$y_p' = -Ke^{-x}\cos x - Ke^{-x}\sin x - Me^{-x}\sin x + Me^{-x}\cos x$

$y_p'' = Ke^{-x}\cos x + Ke^{-x}\sin x + Ke^{-x}\sin x - Ke^{-x}\cos x + Me^{-x}\sin x - Me^{-x}\cos x - Me^{-x}\cos x - Me^{-x}\sin x$

$= 2Ke^{-x}\sin x - 2Me^{-x}\cos x$

Putting these values in equation (1)

$2Ke^{-x}\sin x - 2Me^{-x}\cos x - 6Ke^{-x}\cos x - 6Ke^{-x}\sin x - 6Me^{-x}\sin x + 6Me^{-x}\cos x + 9Ke^{-x}\cos x + 9Me^{-x}\sin x = 50e^{-x}\cos x$

$\Rightarrow (2K - 6K - 6M + 9M)e^{-x}\sin x + (-2M - 6K + 6M + 9K)e^{-x}\cos x = 50e^{-x}\cos x$

Equating both sides

$-4K + 3M = 0$ ——(2)

$3K + 4M = 50$ ——(3)

Solving equation (2) & (3) $K = 6$ $M = 8$

So, $y_p = e^{-x}(6\cos x + 8\sin x)$

Hence, the general solution

$y = y_c + y_p$

$\Rightarrow y = (C_1 + C_2 x)e^{-3x} + e^{-x}(6\cos x + 8\sin x)$

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